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Linear estimate of the number of limit cycles for a class of non-linear systems ⋆

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Abstract

A dynamic system has a finite number of limit cycles. However, finding the upper bound of the number of limit cycles is an open problem for general non-linear dynamical systems. In this paper, we investigated a class of non-linear systems under perturbations. We proved that the upper bound of the number of zeros of the related elliptic integrals of the given system is $7n + 5$ including multiple zeros, which also gives the upper bound of the number of limit cycles for the given system.

1 Introduction

Limit cycle behavior has been observed in many application-oriented dynamical systems. Typical examples of limit cycle behavior include feedback systems [19]; industrial optimization [7]; switched fluid networks for communications, computer systems, and flexible manufacturing [13]; continuous stirred-tank reactors in chemical engineering (CSTR) [17]; machine tool chatters [24], and the generalized Rayleigh-Liénard oscillator [16]. Limit cycles are also observed in the Liénard system, an important class of nonlinear dynamical systems in physics and engineering [6,12,14,16]. They play an important role in the dynamic behavior of many practical systems. The following two aspects are of

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particular interest for charactering the limit cycles of a system qualitatively and quantitatively:

(1) How many limit cycles can a system have?
(2) What is the distribution of the limit cycles?

Research on limit cycle behavior is also important for theoretical advances, and has attracted the interests of many mathematicians ([2,4,5,8–11,15,18,20,21,25]). Consider the following dynamical system

$$\begin{aligned} \dot{x} &= \textstyle\sum_{i+j=0}^{n} a_{ij} x^i y^j = P(x,y) \\ \dot{y} &= \textstyle\sum_{i+j=0}^{n} b_{ij} x^i y^j = Q(x,y) \end{aligned} \tag{E_n}$$

where $(x, y) \in \mathbb{R}^2, a_{ij}, b_{ij} \in \mathbb{R}$. The second part of the Hilbert's 16^{th} problem asks the uniformly bounded upper bound of the number of limit cycles of system (E_n). But this problem has been so difficult that there has been no elegant method to solve it, even for $n = 2$. In 1977, Arnold [2] posed a simpler problem, which was called the weakened Hilbert's 16^{th} problem or the Hilbert-Arnold problem:

Let H be a real polynomial with $\deg H = n$ and f be a polynomial with $\deg f = m$. What is the upper bound of the number of zeros of the following integral?

$$I(h) = \iint\limits_{H \le h} f\, dxdy$$

In this paper, we consider a class of non-linear systems of the form

$$\begin{aligned} \dot{x} &= y + \varepsilon P(x,y), \\ \dot{y} &= -(x^5 + bx^3 + x) + \varepsilon Q(x,y) \end{aligned} \tag{1}$$

where $\varepsilon > 0$ is small, both P and Q are polynomials in x and y. So, the related elliptic integrals can be expressed as the form

$$I(h) = \oint\limits_{\Gamma_h} Q dx + P dy = \oint\limits_{\Gamma_h} \sum_{k+j \le n} a_{kj} x^{2k} y^j dx \tag{A}$$

where Γ_h is the level set $\{H(x,y) = h\}$. For $\varepsilon = 0$, system (1) has a first integral of the form

$$H(x,y) = \frac{y^2 + x^2}{2} + \frac{b}{4} x^4 + \frac{1}{6} x^6 \tag{2}$$

whose level curves have three different types as shown in Fig. 1. They are called the cases of double eight-loops, double-cusps loop, and global center, respectively.

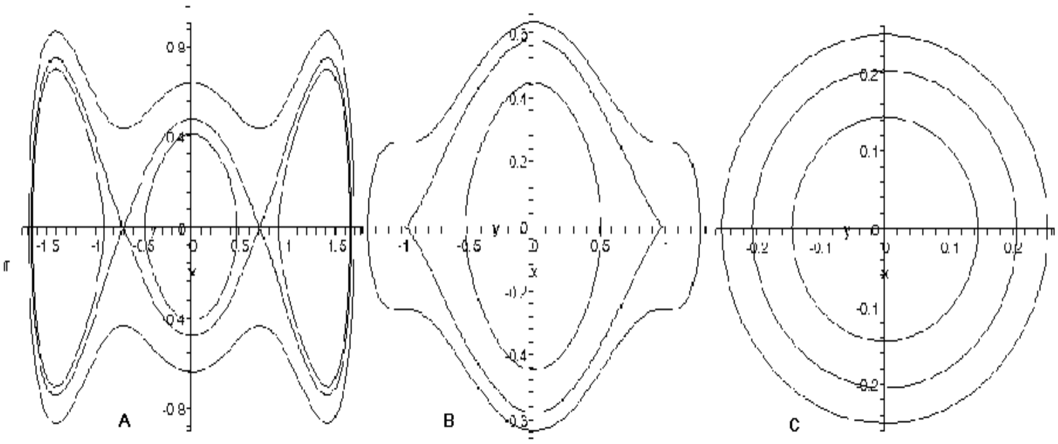

Fig. 1. Level curves of $H(x, y)$.

For system (1), Zang et al. proved in [26] that the Hopf cyclicity is two for $P(x, y) = 0, \deg Q(x, y) = 5$ and $b = -5/2$. Using the bifurcation theory and qualitative analysis, they also investigated the limit cycles bifurcated from homoclinic and heteroclinic loops; Zhang [23] investigated the Abelian integrals of the system for $b < -2$, which is the case (A) in Fig. 1.

The focus of this paper is to investigate the number of zeros of the Abelian integrals of system (1) for $b \geq -2$, i.e., the cases (B) and (C) in Fig.1. The paper is organized as follows. We derive the algebraic structure of the Abelian integrals in Section 2. In Section 3, we first prove our result for the case $b > -2$; and then give the result for the case $b = -2$. The basic idea is to convert the estimate of the number of zeros of the related elliptic integrals into that of a polynomial in h through the Picard-Fuchs equations and Ricatti equation. Finally, some conclusions are drawn in Section 4.

2 Algebraic structure of the Abelian integrals

In this section we will give some relationships that are fundamental for estimating the number of zeros of the Abelian integrals for the case of $b \geq -2$. Some results for the case of $b < -2$ can be found in [23].

Let $I_{k,j} = \oint_{\Gamma_h} x^{2k} y^l dx, I_0 = I_{01}, I_1 = I_{11}$, and $I_2 = I_{21}$. Then, the following statement holds.

Lemma 1 *For $d \geq 3 (k+l = d)$, I_{kl} can be expressed as the linear combination of I_{ij} (where $i + j = d - 1,\ d - 2$) and hI_{ij} (where $i + j = d - 2; i = 0, 1, 2$).*

PROOF. It follows from $H = h$ that

$$\frac{1}{2}I_{k,l+2} + \frac{1}{2}I_{k+1,l} + \frac{b}{4}I_{k+2,l} + \frac{1}{6}I_{k+3,l} = hI_{k,l} \tag{3}$$

and

$$ydy + (x + bx^3 + x^5)dx = 0 \tag{4}$$

Multiplying (4) by $x^{2k-5}y^l$ and integrating over Γ_h give by part

$$I_{kl} = \frac{2k-5}{l+2}I_{k-3,l+2} - I_{k-2,l} - bI_{k-1,l} \tag{5}$$

Combining Eqs. (3) and (5) gives the following algebraic system

$$\mathbf{AJ} = \mathbf{B},\ \mathbf{A} = \begin{pmatrix} \mathbf{A}_5 & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_{d-4} \end{pmatrix} \tag{6}$$

where $\mathbf{J} = (I_{0,d}, I_{1,d-1}, \cdots, I_{d,0})^T, \mathbf{E}_{d-4}$ is a unit matrix of order $d-4$, and

$$\mathbf{A}_5 = \begin{pmatrix} (6d+2)/d & 0 & b & 0 & 0 \\ 0 & 6d/(d-1) & 0 & b & 0 \\ 0 & 0 & (6d-2)/(d-2) & 0 & b \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $\det \mathbf{A}_5 \neq 0$ and matrix $\mathbf{B}$ consists of only integrals I_{ij} (where $i + j = d-1, d-2$) and hI_{ij} (where $i + j = d-2, i = 0, 1, 2$), the statement in the lemma is true. This completes the proof. ■

The straightforward computing by induction yields the following lemma.

Lemma 2 *For $n \geq 3$ the Abelian integral $I(h), h \geq 0$, can be expressed as the following form*

$$I(h) = \alpha(h)I_0 + \beta(h)I_1 + \gamma(h)I_2 \tag{7}$$

where α, β, γ *are polynomials in* h *with* $\deg\alpha = \left[\frac{n-1}{2}\right]$, $\deg\beta = \left[\frac{n-2}{2}\right]$, *and* $\deg\gamma = \left[\frac{n-3}{2}\right]$. *For* $n = 0, \alpha = \beta = \gamma = 0$; *for* $n = 1, \deg\alpha = \beta = \gamma = 0$; *for* $n = 2, \deg\alpha = \deg\beta = \gamma = 0$.

From [22], we have the following lemma.

Lemma 3 *The integrals* I_0, I_1, *and* I_2 *satisfy the following Picard-Fuchs equations*

$$\mathbf{TJ} = (3h\mathbf{E} + \mathbf{S})\mathbf{J}' \tag{8}$$

where $\mathbf{E}$ *is the identity matrix of order 3,* $\mathbf{J} = (I_0, I_1, I_2)^T$, *and*

$$\mathbf{T} = \begin{pmatrix} 2 & 0 & 0 \\ b/4 & 3 & 0 \\ (4-b^2)/4 & 3b/4 & 4 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} 0 & -1 & -b/4 \\ 0 & b/4 & (b^2-4)/4 \\ 0 & (4-b^2)/4 & (5b-b^3)/4 \end{pmatrix}$$

3 Number of zeros of the Abelian integrals

Now we are ready to give a linear estimate of the number of zeros of the abelian integrals. Differentiating both sides of equation (8) with respect to h gives

$$(\mathbf{T} - 3\mathbf{E})\mathbf{J}' = (3h\mathbf{E} + \mathbf{S})\mathbf{J}'' \tag{9}$$

We now consider different cases for Eq. (9) in the following subsections.

3.1 The case of global center: $b > -2$ *and* $3b^2 - 16 \neq 0$

For $3b^2 - 16 \neq 0$, using the special form of $\mathbf{T} - 3\mathbf{E}$ yields

$$I_2''(h) = \frac{12bh}{16-3b^2} I_0''(h) + \frac{48h}{16-3b^2} I_1'(h) \tag{10}$$

Let $Z=\frac{3b}{4}I_1+I_2$. Then, Eqs. (8) and (9) show that for $\mathbf{J}_2=(I_0,I_1,Z)^T$

$$\mathbf{J}_2=\mathbf{D}\mathbf{J}_2',\ \mathbf{D}=\begin{pmatrix} d_{00} & d_{01} & d_{02}\\ d_{10} & d_{11} & d_{12}\\ d_{20} & d_{21} & d_{22}\end{pmatrix} \tag{11}$$

and

$$G(h)\mathbf{J}_2''=\mathbf{D}^{-1}(\mathbf{E}-\mathbf{D}')\mathbf{J}_2'=\mathbf{F}\begin{pmatrix} I_0'\\ Z'\end{pmatrix},\ \mathbf{F}=\begin{pmatrix} f_{00} & f_{02}\\ f_{10} & f_{12}\\ f_{20} & f_{22}\end{pmatrix} \tag{12}$$

where $\mathrm{G}(h)=12\,h\,(-144\,h^2-72\,b\,h+3\,b^2+12\,b^3\,h-16)$,

$$\begin{aligned}
&d_{00}=\frac{3}{2}\,h, && d_{01}=-\frac{1}{2}+\frac{3}{32}\,b^2, && d_{02}=-\frac{1}{8}\,b,\\
&d_{10}=-\frac{1}{8}\,b\,h, && d_{11}=h-\frac{9}{128}\,b^3+\frac{3}{8}\,b, && d_{12}=\frac{3}{32}\,b^2-\frac{1}{3},\\
&d_{20}=\frac{3}{128}\,b^2\,h-\frac{3}{8}\,h, && d_{21}=\frac{27}{2048}\,b^4-\frac{9}{64}\,b^2+\frac{3}{8}, && d_{22}=-\frac{9}{512}\,b^3+\frac{5}{32}\,b+\frac{3}{4}\,h
\end{aligned}$$

and

$$\begin{aligned}
&f_{00}=576\,h^2+(-36\,b^3+192\,b)\,h, && f_{02}=12\,b^2-48\,b\,h-64,\\
&f_{10}=-144\,b\,h^2+(36\,b^2-192)\,h, && f_{12}=48\,h\,(b^2-4),\\
&f_{20}=(36\,b^2-576)\,h^2+(27\,b^3-144\,b)\,h, && f_{22}=12\,h\,(-48\,h+3\,b^3-16\,b).
\end{aligned}$$

So, the integral $I(h)$ can be expressed as the form

$$I(h)=\alpha(h)I_0+(\beta(h)-\frac{3b}{4}\gamma(h))I_1+\gamma(h)Z \tag{13}$$

and

$$I'(h)=\alpha_1(h)I_0'+\beta_1(h)I_1'+\gamma_1(h)Z' \tag{14}$$

where

$$\alpha_1 = \alpha' d_{00} + (\beta' - \frac{3}{4}b\gamma')d_{10} + \gamma' d_{20} + \alpha,$$
$$\beta_1 = \alpha' d_{01} + (\beta' - \frac{3}{4}b\gamma')d_{11} + \gamma' d_{21} + \beta - \frac{3}{4}b\gamma,$$
$$\gamma_1 = \alpha' d_{02} + (\beta' - \frac{3}{4}b\gamma')d_{12} + \gamma' d_{22} + \gamma$$

with $\deg \alpha_1 \leq \left[\frac{n-1}{2}\right]$, $\deg \beta_1 \leq \left[\frac{n-2}{2}\right]$, and $\deg \gamma_1 \leq \left[\frac{n-3}{2}\right]$.

Let $\Sigma = \{h | h > 0, \}, \mathcal{N} = \{h | \beta_1(h) = 0 \text{ or } h = h_2, h \in \Sigma\}$. Then, from [23] we obtain the following lemma.

Lemma 4 *For* $h \in \Sigma \setminus \mathcal{N}$

$$\left(\frac{I'(h)}{\beta_1(h)}\right)' = \frac{M(h)}{G(h)\beta_1^2(h)} \tag{15}$$

where

$$M(h) = \alpha_2(h) I_0' + \beta_2(h) Z' \tag{16}$$

with $\deg \alpha_2 \leq \left[\frac{n-1}{2}\right] + \left[\frac{n-2}{2}\right] + 2$, $\deg \beta_2 \leq \left[\frac{n-1}{2}\right] + \left[\frac{n-2}{2}\right] + 1$, and $\alpha_2(0) = 0$.

Lemma 5 *Let p, q, and r be the number of zeros of $I'(h), M(h), \beta_1(h)$ in the interval $(0, +\infty)$, respectively. Then*

$$p \leq q + r + 1$$

PROOF. Let $a_1, a_2, \cdots, a_r$ be the zeros of β_1 in the interval $(0, +\infty)$ with $0 = a_0 < a_1 < a_2 < \cdots < a_r < +\infty$, as shown in Fig. 2.

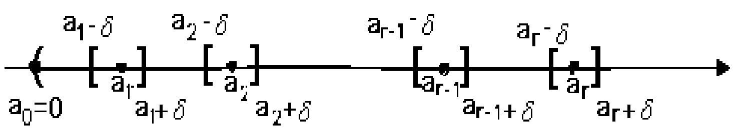

Fig. 2. The distribution of zeros of β_1.

Then, for each $a_j, j = 1, \cdots, r$, there is a $\delta > 0$ such that except a_j there are no zeros of I', M and β_1 in the interval $[a_j - \delta, a_j + \delta]$.

Now, suppose a_j is the zero of I', M and β_1 with multiplication l_j, m_j, and k_j, respectively. If $k_j = l_j$, then $m_j - 2l_j \geq 0$, which implies that $m_j \geq l_j$; if $k_j \neq l_j$, then $l_j - k_j - 1 = m_j - 2k_j$, which also implies $m_j \geq l_j$.

On the other hand, let p_0, p_j, and p_r denote the number of zeros of I' in the intervals $(a_0, a_1-\delta), (a_j+\delta, a_{j+1}-\delta)$, and $(a_r+\delta, +\infty)$, respectively, and let q_0, q_j, and q_r denote the number of zeros of M in the intervals $(a_0, a_1-\delta), (a_j+\delta, a_{j+1}-\delta)$, and $(a_r+\delta, +\infty)$, respectively, where $j = 1, \cdots, r-1$. Then, Eq. (15) implies $p_j \leq q_j + 1, j = 0, \cdots, r$. Therefore,

$$p = \sum_{j=0}^{r} p_j + \sum_{j=1}^{r} l_j \leq \sum_{j=0}^{r} (q_j+1) + \sum_{j=1}^{r} m_j = [q_0 + \sum_{j=1}^{r} (q_j + m_j)] + r + 1 = q + r + 1.$$

■

Lemma 6 *Let $W(h) = M(h)/I_0'$. Then, we can obtain the Riccati equation*

$$G\beta_2 W' = R_2 + (G\beta_2' + \beta_2(f_{22} - f_{00}) + 2\alpha_2 f_{02})W - f_{02}W^2 \qquad (17)$$

Furthermore, if letting s and t respectively denote the number of zeros of R_2 and β_2 in the interval $(0, +\infty)$, then we have

$$p \leq r + s + t + 2$$

PROOF. Using interval $(0, +\infty)$ instead of $(0, h_2)$ in [23], the straightforward computing and Lemma 5 give these results.

■

Let $\#\{I(h), h \in (m, n)\}$ denote the number of zeros of $I(h)$ with respect to h in the interval (m, n). Then, Eq. (14) implies

$$\#\{I(h), h \in (0, +\infty)\} \leq \#\{I'(h), h \in (0, +\infty)\} + 1$$

Since $I(0) = R(0) = 0$ we have

$$\#\{I(h), h \in (0, +\infty)\} \leq \#\{I'(h), h \in (0, +\infty)\} + 1 - 1 \leq r + (s-1) + t + 2 \leq \frac{7n+3}{2}$$

3.2 The case of global center: $b > -2$ and $3b^2 - 16 = 0$

The equality $3b^2 - 16 = 0$ means $b = \dfrac{4\sqrt{3}}{3}$.

In this section, we are going to use equations $\mathbf{J}_2 = (I_0, Z)^T$,

$$I_1'' = \frac{\sqrt{3}}{3} I_0'' \qquad (18)$$

and

$$G(h)\mathbf{J}_2'' = \begin{pmatrix} f_{00} & f_{02} \\ f_{20} & f_{22} \end{pmatrix} = \mathbf{F}_2\mathbf{J}_2' \tag{19}$$

instead of equations (10) and (12). We will also use equations (13) and (14) with $b = \dfrac{4\sqrt{3}}{3}$. Similar to the result given in section 3.1, the following result is obtained

$$\#\{I(h), h \in (0, +\infty)\} \leq \frac{7n+3}{2}$$

3.3 The case of double-cusps loop: $b = -2$.

In this case, $h = h_2 = \dfrac{1}{6}$ corresponds to the double-cusps loop. Since $b = -2$, equations (10) through (14) hold. Then, using $\Sigma_1 = \{h|h > 0, h = h_2\}$ and $\mathcal{N}_1 = \{h|\beta_1(h) = 0\,, h \in \Sigma_1\}$ instead of Σ and $\mathcal{N}$, we can obtain

$$\#\{I(h), h \in (0, +\infty)\} \leq \#\{I'(h), h \in (0, h_2)\} + \#\{I'(h), h \in (h_2, +\infty)\} \leq 7n+5$$

3.4 Main Result

The results given in subsections 3.1 through 3.3 are summarised in the following theorem.

Theorem 1 *Suppose that $I(h)$ is defined in Eq. (A) and $a_{k,j} \in \mathbb{R}$. The following relation holds*

$$\#\{I(h), h \in (0, +\infty)\} \leq 7n + 5.$$

4 Conclusions

In this paper, we have investigated the number of zeros of the Abelian integrals for a class of non-linear dynamical systems for the case $b \geq -2$. We have proved that the related elliptic integrals have at most $7n + 5$ zeros including multiple zeros. Through the results presented in this paper and [23], we can obtain the upper bound of the number of limit cycles bifurcated from the periodical orbits for each $b \in \mathbb{R}$.

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